Classical BRST charge and observables in reducible gauge theories

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March, 2012

Abstract

The general solution to the master equation for the classical BRST charge is found in the case of reducible gauge theories. The BRST observables are constructed.

1 Introduction

The modern quantization method for gauge theories is based on the Becchi-Rouet-Stora-Tyutin (BRST) symmetry [1, 2]. In the framework of the canonical formalism this symmetry is generated by the BRST charge. If the quantum BRST charge exists it is essentially determined by the corresponding classical one. The BRST charge Ω is defined as a solution to the master equation

$$\{\Omega, \Omega\} = 0 \tag{1}$$

with certain boundary conditions. The BRST construction in the case of reducible gauge theories was given in [3, 4]. The global existence of the BRST charge was proved in [5]. However, the problem of finding the BRST charge and observables has not been solved.

In this paper we give the general solution to equation (1) for the general case of reducible gauge theories. To this aim, we construct a new coordinate system in the extended phase space and transform the equation by changing

variables. Then it can be solved by an iterative method. In the framework of the Lagrangian formalism a similar equation was solved in [6]. We also give the general solution to the equation determining the BRST observables.

The paper is organized as follows. In section 2, we review the BRST construction and derive some auxiliary equations. In section 3, we introduce new variables and construct a generalized inverse of the Koszul-Tate gifferential operator δ. With respect to the new variables, both δ and its generalized inverse take a standart form. The general solution to the master equation is given in section 4. The BRST observables are described in section 5.

In what follows Grassman parity and ghost number of a function X are denoted by $\epsilon(X)$ and $\mathrm{gh}(X)$, respectively. The Poisson superbracket in phase space $\Gamma = (P_i, Q^i), \epsilon(P_i) = \epsilon(Q^i)$, is given by

$$\{X, Y\} = \frac{\partial X}{\partial Q^i}\frac{\partial Y}{\partial P_i} - (-1)^{\epsilon(X)\epsilon(Y)}\frac{\partial Y}{\partial Q^i}\frac{\partial X}{\partial P_i}.$$

Derivatives with respect to generalized momenta P_i are always understood as left-hand, and those with respect to generalized coordinates Q^i as right-hand ones.

2 Master equation for the BRST charge

Let $\xi_a, a = 1, \ldots, 2m$, be the phase space coordinates, and let $G_{a_0}, a_0 = 1, \ldots, m_0 \leq 2m$, be the first class constraints which satisfy the following Poisson brackets

$$\{G_{a_0}, G_{b_0}\} = F^{c_0}_{a_0 b_0} G_{c_0},$$

where $F^{c_0}_{a_0 b_0}$ are phase space functions. The constraints are supposed to be of definite Grassmann parity ϵ_{a_0}, $\epsilon(G_{a_0}) = \epsilon_{a_0}$.

We shall consider a reducible theory of L-th order. That is, there exist phase space functions

$$Z^{a_k}_{a_{k+1}}, \qquad k = 0, \ldots, L-1, \qquad a_k = 1, \ldots, m_k,$$

such that at each stage the Z's form a complete set,

$$Z^{a_k}_{a_{k+1}} \lambda^{a_{k+1}} \approx 0 \Rightarrow \lambda^{a_{k+1}} \approx Z^{a_{k+1}}_{a_{k+2}} \lambda^{a_{k+2}}, \qquad k = 0, \ldots, L-2,$$

$$Z^{a_{L-1}}_{a_L} \lambda^{a_L} \approx 0 \Rightarrow \lambda^{a_L} \approx 0.$$

$$G_{a_0} Z^{a_0}_{a_1} = 0, \qquad Z^{a_k}_{a_{k+1}} Z^{a_{k+1}}_{a_{k+2}} \approx 0, \qquad k = 0, \ldots, L-2. \tag{2}$$

The weak equality $\approx$ means equality on the constraint surface

$$\Sigma : \qquad G_{a_0} = 0.$$

Following the BRST method the ghost pairs $(\mathcal{P}_{a_k}, c^{a_k}), k = 0, \ldots, L,$ are introduced

$$\epsilon(\mathcal{P}_{a_k}) = \epsilon(c^{a_k}) = \epsilon_{a_k} + k + 1, \qquad -\mathrm{gh}(\mathcal{P}_{a_k}) = \mathrm{gh}(c^{a_k}) = k + 1.$$

The BRST charge Ω is defined as a solution to the equations (1),

$$\epsilon(\Omega) = 1, \qquad \mathrm{gh}(\Omega) = 1, \tag{3}$$

and the boundary conditions

$$\left.\frac{\partial \Omega}{\partial c^{a_0}}\right|_{c=0} = G_{a_0}, \qquad \left.\frac{\partial^2 \Omega}{\partial \mathcal{P}_{a_{k-1}} \partial c^{a_k}}\right|_{\mathcal{P}=c=0} = Z^{a_{k-1}}_{a_k}.$$

One can write

$$\Omega = \Omega^{(1)} + M, \qquad M = \sum_{n \geq 2} \Omega^{(n)}, \qquad \Omega^{(n)} \sim c^n, \tag{4}$$

$$\Omega^{(1)} = G_{a_0} c^{a_0} + \sum_{k=1}^{L} (\mathcal{P}_{a_{k-1}} Z^{a_{k-1}}_{a_k} + N_{a_k}) c^{a_k}, \tag{5}$$

where $N_{a_1} = 0$ and N_{a_k}, $k > 1$, only involves $\mathcal{P}_{a_s}$, $s \leq k-2$. Eq. (3) implies

$$N_{a_k}|_{\mathcal{P}=0} = 0, \qquad M|_{\mathcal{P}=0} = 0. \tag{6}$$

Let V be the space of the formal power series in $(\xi, \mathcal{P}, c)$ which vanish on Σ at $\mathcal{P} = 0$. For $X, Y \in V$ we have

$$XY \in V, \qquad \{X, Y\} \in V,$$

and therefore V is a Poisson algebra. One easily verifies that Ω, as well as $\{\Omega, \Omega\}$, belong to V.

The bracket $\{.,.\}$ splits as

$$\{X,Y\} = \{X,Y\}_\xi + \{X,Y\}_\diamond - (-1)^{\epsilon(X)\epsilon(Y)}\{Y,X\}_\diamond,$$

where $\{.,.\}_\xi$ refers to the Poisson bracket in the original phase space and

$$\{X,Y\}_\diamond = \sum_{k=0}^{L} \frac{\partial X}{\partial c^{a_k}} \frac{\partial Y}{\partial \mathcal{P}_{a_k}}.$$

Let $\delta : V \to V$ be defined by

$$\delta = \{\Omega^{(1)}, .\}_\diamond = G_{a_0} \frac{\partial}{\partial \mathcal{P}_{a_0}} + \sum_{k=1}^{L} (\mathcal{P}_{a_{k-1}} Z^{a_{k-1}}_{a_k} + N_{a_k}) \frac{\partial}{\partial \mathcal{P}_{a_k}}.$$

Substituting (4) in (1) one obtains

$$\delta\Omega^{(1)} = 0, \tag{7}$$

$$\delta M = D, \tag{8}$$

where

$$-D = \frac{1}{2}F + AM + \frac{1}{2}\{M,M\}, \qquad F = \{\Omega^{(1)}, \Omega^{(1)}\}_\xi,$$

and the operator A is given by

$$AX = \{\Omega^{(1)}, X\}_\xi - (-1)^{\epsilon(X)}\{X, \Omega^{(1)}\}_\diamond.$$

Eq. (7) is equivalent to

$$\delta^2 = 0.$$

Let $\{X,Y\}^{(1)}$ denote the first order contribution in the ghosts to the bracket $\{X,Y\}$. The Jacobi identity $\{\Omega^{(1)}, \{\Omega^{(1)}, \Omega^{(1)}\}\} = 0$ implies

$$\{\Omega^{(1)}, \{\Omega^{(1)}, \Omega^{(1)}\}\}^{(1)} = 0.$$

Since $\{\Omega^{(1)}, \Omega^{(1)}\} = 2\delta\Omega^{(1)} + O(c^2)$, we have $\{\Omega^{(1)}, \delta\Omega^{(1)}\}^{(1)} = 0$, from which it follows that

$$\delta^2\Omega^{(1)} = \{\delta\Omega^{(1)}, \Omega^{(1)}\}_\diamond. \tag{9}$$

If (7) holds, then $\{\Omega, \Omega\} = R$, where R is the difference between the left-hand and right-hand sides of (8),

$$R = \delta M + \frac{1}{2}F + AM + \frac{1}{2}\{M, M\}.$$

From the Jacobi identity $\{\Omega, \{\Omega, \Omega\}\} = 0$ it follows that $\{\Omega, R\} = 0$, or equivalently

$$\delta R + AR + \{M, R\} = 0. \tag{10}$$

Here we have used the relation

$$\{\Omega^{(1)}, \, .\, \} = \delta + A.$$

3 Generalized inversion of δ

For $k = L - 1$, eq. (2) reads

$$Z^{a_{L-2}}_{a'_{L-1}} Z^{a'_{L-1}}_{a_L} + Z^{a_{L-2}}_{a''_{L-1}} Z^{a''_{L-1}}_{a_L} \approx 0, \tag{11}$$

where a'_{L-1}, a''_{L-1} are increasing indexes, such that $a'_{L-1} \cup a''_{L-1} = a_{L-1}$, $|a'_{L-1}| = |a_L|$ and $\operatorname{rank} Z^{a'_{L-1}}_{a_L} = |a_L|$. For an index set $i = \{i_1, i_2, \ldots, i_n\}$, we denote $|i| = n$. From (11) it follows that $\operatorname{rank} Z^{a_{L-2}}_{a_{L-1}} = |a_{L-1}| - |a_L| = |a''_{L-1}|$ and $\operatorname{rank} Z^{a_{L-2}}_{a''_{L-1}} = |a''_{L-1}|$.

One can split the indexes a_{L-2} as $a_{L-2} = a'_{L-2} \cup a''_{L-2}$, such that $|a'_{L-2}| = |a''_{L-1}|$, and $\operatorname{rank} Z^{a'_{L-2}}_{a''_{L-1}} = |a''_{L-1}|$. For $k = L - 2$, eq. (2) implies

$$Z^{a_{L-3}}_{a'_{L-2}} Z^{a'_{L-2}}_{a''_{L-1}} + Z^{a_{L-3}}_{a''_{L-2}} Z^{a''_{L-2}}_{a''_{L-1}} \approx 0.$$

From this it follows that

$$\operatorname{rank} Z^{a_{L-3}}_{a''_{L-2}} = \operatorname{rank} Z^{a_{L-3}}_{a_{L-2}} = |a_{L-2}| - |a''_{L-1}| = |a''_{L-2}|.$$

Proceeding inductively, one can obtain a set of nonsingular matrixes $Z^{a'_{k-1}}_{a''_k}$, $k = 2, \ldots, L$, and a set of matrixes $Z^{a_{k-1}}_{a''_k}$, $k = 1, \ldots, L$, such that

$$\operatorname{rank} Z^{a_{k-1}}_{a''_k} = Z^{a_{k-1}}_{a_k} = |a''_k|.$$

Eq. (2) implies

$$G_{a'_0} Z^{a'_0}_{a''_1} + G_{a''_0} Z^{a''_0}_{a''_1} = 0, \tag{12}$$

where the indexes a_0 split as $a_0 = a'_0 \cup a''_0$, such that $|a'_0| = |a''_1|$, and rank $Z^{a'_0}_{a''_1} = |a''_1|$. From (12) it follows that $G_{a''_0}$ are independent. We assume that $G_{a''_0}$ satisfy the regularity conditions. It means that there are some functions $F_\alpha(\xi)$, $\alpha \cup a''_0 = a$, such that $(F_\alpha, G_{a''_0})$ can be locally taken as new coordinates in the phase space.

For $a''_{k+1} \in a_{k+1}$ we define an embedding $f(a''_{k+1}) = a''_{k+1} \in a_k$, $k = 0, \ldots, L-2$, $f(a''_L) = a_L \in a_{L-1}$. Let α_k be defined by $a_k = f(a''_{k+1}) \cup \alpha_k$. Since $|a''_k| = |\alpha_k|$, one can write $\alpha_k = g(a''_k)$ for some function g.

Lemma. [6] The nilpotent operator δ is reducible to the form

$$\delta = \xi'_{a''_0} \frac{\partial}{\partial \mathcal{P}'_{g(a''_0)}} + \sum_{k=1}^{L} \mathcal{P}'_{f(a''_k)} \frac{\partial}{\partial \mathcal{P}'_{g(a''_k)}}, \tag{13}$$

by the change of variables: $(\xi, \mathcal{P}_{a_0}, \ldots, \mathcal{P}_{a_L}) \to (\xi', \mathcal{P}'_{a_0}, \ldots, \mathcal{P}'_{a_L})$,

$$\xi'_\alpha = F_\alpha, \qquad \xi'_{a''_0} = G_{a''_0},$$

$$\mathcal{P}'_{f(a''_{k+1})} = \mathcal{P}_{a_k} Z^{a_k}_{a''_{k+1}} + N_{a''_{k+1}}, \qquad \mathcal{P}'_{\alpha_k} = \mathcal{P}_{g^{(-1)}(\alpha_k)}, \tag{14}$$

$$\mathcal{P}'_{a_L} = \mathcal{P}_{a_L},$$

where $k = 0, \ldots, L-1$, $g(a''_L) = a_L$.

Eqs. (14) are solvable with respect to the original variables and can be represented as

$$\xi_a = \xi_a(\xi'), \qquad \mathcal{P}_{a_k} = \mathcal{P}_{a_k}(\xi'_{a_0}, \mathcal{P}'_{a_0}, \ldots, \mathcal{P}'_{a_k}), \qquad k = 0, \ldots, L.$$

Here we have used the fact that the $\mathcal{P}_{a_k}$ depends only on the functions $\mathcal{P}'_{a_s}$ with $s \leq k$. Assume that the functions $\xi_a(\xi')$ have been constructed. Then from (14) it follows that

$$\mathcal{P}_{a'_k} = (\mathcal{P}'_{f(a''_{k+1})} - \mathcal{P}'_{g(a''_k)} Z'^{a''_k}_{a''_{k+1}} - N'_{a''_{k+1}})(Z'^{(-1)})^{a''_{k+1}}_{a'_k}, \qquad \mathcal{P}_{a''_k} = \mathcal{P}'_{g(a''_k)},$$

$$\mathcal{P}_{a_L} = \mathcal{P}'_{a_L}, \tag{15}$$

where $k = 0, \ldots, L-1$,

$$Z'^{a_k}_{a_{k+1}}(\xi') = Z^{a_k}_{a_{k+1}}(\xi), \qquad N'_{a_k}(\xi', \mathcal{P}'_{a_0}, \ldots, \mathcal{P}'_{a_{k-2}}) = N_{a_k}(\xi, \mathcal{P}_{a_0}, \ldots, \mathcal{P}_{a_{k-2}}).$$

Let n be the counting operator

$$n = \xi'_{a''_0} \frac{\partial}{\partial \xi'_{a''_0}} + \mathcal{P}'_{g(a''_0)} \frac{\partial}{\partial \mathcal{P}'_{g(a''_0)}} + \sum_{s=1}^{L} \left(\mathcal{P}'_{f(a''_s)} \frac{\partial}{\partial \mathcal{P}'_{f(a''_s)}} + \mathcal{P}'_{g(a''_s)} \frac{\partial}{\partial \mathcal{P}'_{g(a''_s)}} \right),$$

and let

$$\sigma = \mathcal{P}'_{g(a''_0)} \frac{\partial}{\partial \xi'_{a''_0}} + \sum_{s=1}^{L} \mathcal{P}'_{g(a''_s)} \frac{\partial}{\partial \mathcal{P}'_{f(a''_s)}}.$$

One can directly verify that

$$\sigma^2 = 0, \qquad \delta\sigma + \sigma\delta = n, \qquad n\delta = \delta n, \qquad n\sigma = \sigma n. \tag{16}$$

With respect to the new coordinate system the condition $X \in V$ implies

$$X|_{\xi'_{a''_0} = \mathcal{P}' = 0} = 0.$$

The space V splits as

$$V = \bigoplus_{m \geq 1} V_m \tag{17}$$

with $nX = mX$ for $X \in V_m$. Hence the operator $n : V \to V$ is invertible. The inverse of n is given by

$$n^{(-1)} = \int_0^\infty \exp(-tn) dt.$$

One easily verifies that $\delta^+ = \sigma n^{(-1)}$ is a generalized inverse of δ:

$$\delta\delta^+\delta = \delta, \qquad \delta^+\delta\delta^+ = \delta^+. \tag{18}$$

From (16) and (17) it follows that for any $X \in V$,

$$X = \delta^+\delta X + \delta\delta^+ X. \tag{19}$$

4 Solution of the master equation

Let S denote the left-hand side of (7), $S = \delta\Omega^{(1)}$. Identity (9) takes the form

$$\delta S = \{S, \Omega^{(1)}\}_\diamond. \tag{20}$$

It is straightforward to check that

$$S = \delta N + Q + BN,$$

where

$$N = \sum_{k=2}^{L} N_{a_k} c^{a_k}, \qquad Q = \sum_{k=2}^{L} \mathcal{P}_{a_{k-2}} Z^{a_{k-2}}_{a_{k-1}} Z^{a_{k-1}}_{a_k} c^{a_k},$$

$B : V \to V$ is defined by $B = 0$, if $L \leq 2$, and otherwise

$$BX = \sum_{k=3}^{L} \frac{\partial X}{\partial c^{a_{k-1}}} Z^{a_{k-1}}_{a_k} c^{a_k}.$$

Eq. (7) can be written in the form

$$\delta N + Q + BN = 0. \tag{21}$$

Changing variables $(\xi, \mathcal{P}) \to (\xi', \mathcal{P}')$, we get

$$\delta N' + Q' + B'N' = 0. \tag{22}$$

Here and in what follows, for any $X(\xi, \mathcal{P}, c)$ we denote by X' the function

$$X'(\xi', \mathcal{P}', c) = X(\xi, \mathcal{P}, c).$$

Applying $\delta\delta^+$ to (22) and using (18) we have

$$\delta N' + \delta\delta^+(Q' + B'N') = 0,$$

from which it follows that

$$N' + \delta^+(Q' + B'N') = Y', \tag{23}$$

where

$$Y' = \sum_{k=2}^{L} Y'_{a_k}(\xi', \mathcal{P}')c^{a_k}, \qquad \delta Y' = 0, \qquad \epsilon(Y') = 1, \qquad \mathrm{gh}(Y') = 1.$$

Solving (23), we get

$$N' = (I + \delta^+ B')^{(-1)}(Y' - \delta^+ Q'), \tag{24}$$

where I is the identity map, and

$$(I + \delta^+ B)^{(-1)} = \sum_{m \geq 0} (-1)^m (\delta^+ B)^m.$$

It remains to show that (24) satisfies (22). We shall use the approach of [7]. With respect to the new coordinate system eq. (20) takes the form

$$\delta S' = \{S', \Omega'^{(1)}\}'_\diamond, \tag{25}$$

where

$$S' = \delta N' + Q' + B'N'.$$

If N' is a solution to (23), then

$$\delta^+ N' = \delta^+ Y',$$

since $(\delta^+)^2 = 0$, and

$$\delta^+ S' = \delta^+ \delta N' + \delta^+(Q' + B'N') = 0. \tag{26}$$

Here we have used (19) and (23). Consider eq. (25) and condition (26), where N' is the solution to (23). Applying δ^+ to (25), and using (26), we get

$$S' = \delta^+ \{S', \Omega'^{(1)}\}'_\diamond,$$

from which by iterations it follows that $S' = 0$.

For $2 \leq k \leq L$, denote

$$V\{k\} = \{\Phi \in V \mid \Phi = \Phi_{a_k}(\xi', \mathcal{P}')c^{a_k}\}.$$

One easily verifies that

$$B'V\{k\} \subset V\{k+1\}, \qquad k < L, \qquad B'V\{L\} = 0, \qquad \delta^+ V\{k\} \subset V\{k\},$$

and therefore the series (24) is finite.

Let us now consider eq. (8). Changing variables $(\xi, \mathcal{P}) \to (\xi', \mathcal{P}')$, we get

$$\delta M' = D', \tag{27}$$

where

$$-D' = \frac{1}{2}F' + AM' + \frac{1}{2}\{M', M'\}'.$$

Applying $\delta\delta^+$ to (27), we have

$$\delta M' = \delta\delta^+ D',$$

from which it follows that

$$M' = W' + \delta^+ D', \tag{28}$$

$$W' \in V, \qquad \delta W' = 0, \qquad \epsilon(W') = 1, \qquad \mathrm{gh}(W') = 1. \tag{29}$$

Let $\langle .\,, . \rangle : V^2 \to V$ be defined by

$$\langle X_1, X_2 \rangle = -\frac{1}{2}(I + \delta^+ A)^{(-1)}\delta^+ \left(\{X_1, X_2\}' + \{X_2, X_1\}'\right),$$

where I is the identity map, and

$$(I + \delta^+ A)^{(-1)} = \sum_{m \geq 0} (-1)^m (\delta^+ A)^m.$$

One can rewrite (28) as

$$M' = M'_0 + \frac{1}{2}\langle M', M' \rangle, \tag{30}$$

where

$$M'_0 = (I + \delta^+ A)^{(-1)}(W' - \frac{1}{2}\delta^+ F').$$

Eq. (30) can be iteratively solved as:

$$M' = M'_0 + \frac{1}{2}\langle M'_0, M'_0\rangle + \dots . \tag{31}$$

Changing variables in (10) $(\xi, \mathcal{P}) \to (\xi', \mathcal{P}')$, we get

$$\delta R' + AR' + \{M', R'\}' = 0, \tag{32}$$

where

$$R' = \delta M' + \frac{1}{2}F' + AM' + \frac{1}{2}\{M', M'\}'. \tag{33}$$

To prove that (31) satisfies (27) consider eq. (32) and the condition

$$\delta^+ R' = 0, \tag{34}$$

where M' is the solution to (28). Applying δ^+ to eq. (32) and using (34), we get

$$R' = -\delta^+(AR' + \{M', R'\}'). \tag{35}$$

From (35) by iterations it follows that $R' = 0$.

The solution to (28) satisfies the condition

$$\delta^+ M' = \delta^+ W'.$$

By using (19), we have

$$M' = \delta^+\delta M' + \delta\delta^+ W'.$$

From this, (19) and (29) it follows that

$$M' = \delta^+\delta M' + W'. \tag{36}$$

To check (34), we have

$$\delta^+ R' = \delta^+\delta M' + \delta^+(\frac{1}{2}F' + AM' + \frac{1}{2}\{M', M'\}'),$$

and therefore by (36) and (28), $\delta^+ R' = 0$.

5 BRST observables

Let $\Phi_0(\xi)$ be a first class function

$$\{G_{a_0}, \Phi_0\} \approx 0.$$

A function $\Phi = \Phi(\xi, \mathcal{P}, c)$ is called a BRST-invariant extension of Φ_0 if

$$\Phi = \Phi_0 + \Pi, \qquad \Pi = \sum_{n \geq 1} \Phi^{(n)}, \qquad \Phi^{(n)} \sim c^n, \qquad \mathrm{gh}(\Phi) = 0,$$

$$\{\Omega, \Phi\} = 0. \tag{37}$$

Let $\mathcal{U}$ denote the space of all such extensions. The functions $\Phi_1, \Phi_2 \in \mathcal{U}$ are set to be equivalent if

$$\Phi_1 - \Phi_2 = \{\Omega, \Psi\} \tag{38}$$

for some Ψ. Elements of the corresponding factorspace $\mathcal{U}/\!\sim$ are called the BRST observables.

Let us consider the equation

$$\{\Omega, \Psi\} - \Lambda = 0, \tag{39}$$

where Λ is a given function, $\mathrm{gh}(\Lambda) = 0$, $\{\Omega, \Lambda\} = 0$, and Ψ is an unknown one. The equation implies that $\Psi, \Lambda \in V$, since $\mathrm{gh}(\Psi) = -1$. Let us show that for any $\Lambda \in V$ there exist a solution to (39). One can write (39) in the form

$$\delta\Psi + A\Psi + \{M, \Psi\} - \Lambda = 0. \tag{40}$$

Changing variables from $(\xi, \mathcal{P})$ to $(\xi', \mathcal{P}')$, we get

$$\delta\Psi' + A\Psi' + \{M', \Psi'\}' - \Lambda' = 0. \tag{41}$$

By using (18), one can write

$$\Psi' + \delta^+(A\Psi' + \{M', \Psi'\}' - \Lambda') = \Upsilon', \tag{42}$$

where

$$\Upsilon' \in V, \qquad \delta\Upsilon' = 0, \qquad \mathrm{gh}(\Upsilon') = 1.$$

From (42) it follows

$$\Psi' = (I + \delta^+(A + \mathrm{ad}\, M'))^{(-1)}(\Upsilon' + \delta^+\Lambda'), \tag{43}$$

where $\mathrm{ad}\, M' = \{M', \,.\,\}'$.

Now, let us show that (43) satisfies (41). Denote by Γ left-hand side of (39)

$$\Gamma = \{\Omega, \Psi\} - \Lambda.$$

From the Jacoby identity $\{\Omega, \{\Omega, \Psi\}\} = 0$ and the BRST invariance of Λ it follows that

$$\delta\Gamma + A\Gamma + \{M, \Gamma\} = 0.$$

Changing variables $(\xi, \mathcal{P}) \to (\xi', \mathcal{P}')$, we get

$$\delta\Gamma' + A\Gamma' + \{M', \Gamma'\}' = 0, \tag{44}$$

where

$$\Gamma' = \{\Omega', \Psi'\}' - \Lambda'.$$

It is straightforward to check that if Ψ' satisfies (42) then $\delta^+\Psi' = \delta^+\Upsilon'$, and

$$\delta^+\Gamma' = 0. \tag{45}$$

Consider (44) and (45), where Ψ' satisfies (42). By using (19), we get

$$\Gamma' = -\delta^+(A\Gamma' + \{M', \Gamma'\}'), \tag{46}$$

from which it follows that $\Gamma' = 0$. From definition (38) we conclude that $\Phi_1 \sim \Phi_2$ if and only if $\Phi_1 - \Phi_2 \in V$.

Let us now turn our attention to eq. (37). It can be written in the form

$$\delta\Pi + \mathrm{P} + A\Pi + \{M, \Pi\} = 0, \tag{47}$$

where $\mathrm{P} = \{\Omega, \Phi_0\}$. We note that Π and left-hand side of (47) belong to V. Changing variables from $(\xi, \mathcal{P})$ to $(\xi', \mathcal{P}')$, we get

$$\delta\Pi' + \mathrm{P}' + A\Pi' + \{M', \Pi'\}' = 0. \tag{48}$$

By repeating the same steps as in the case of eq. (41), we obtain the general solution to (48)

$$\Pi' = (I + \delta^+(A + \mathrm{ad}\, M'))^{(-1)}(\mathrm{X}' - \delta^+\mathrm{P}'),$$

$$\mathrm{X}' \in V, \qquad \delta\mathrm{X}' = 0, \qquad \mathrm{gh}(\mathrm{X}') = 0.$$

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